

# Automorphism-Invariant Non-Singular Rings and Modules

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**Abstract.** A ring  $A$  is a right automorphism-invariant right non-singular ring if and only if  $A = S \times T$ , where  $S$  a right injective regular ring and  $T$  is a strongly regular ring which contains all invertible elements of its maximal right ring of quotients. Over a ring  $A$ , each direct sum of automorphism-invariant non-singular right modules is an automorphism-invariant module if and only if the factor ring of the ring  $A$  with respect to its right Goldie radical is a semiprime right Goldie ring.

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**Key words:** automorphism-invariant ring, automorphism-invariant module, injective module, quasi-injective module

## 1. Introduction and preliminaries

All rings are assumed to be associative and with zero identity element; all modules are unitary. A module  $M$  is said to be *automorphism-invariant* if  $M$  is invariant under any automorphism of its injective hull. In [2] Dickson and Fuller studied automorphism-invariant modules, when the underlying ring is a finite-dimensional algebra over a field with more than two elements. In [3, Theorem 16] Er, Singh and Srivastava proved that a module  $M$  is automorphism-invariant if and only if  $M$  is a *pseudo-injective* module, i.e., for any submodule  $X$  of  $M$ , every monomorphism  $X \rightarrow M$  can be extended to an endomorphism of the module  $M$ . Pseudo-injective modules were studied in several papers; e.g., see [8], [14], [3]. Automorphism-invariant modules were studied in several papers; e.g., see [1], [3], [5], [11], [13], [15], [16], [17], [18], [19].

A ring  $A$  is said to be *regular* if every its principal right (left) ideal is generated by an idempotent. A ring  $A$  is said to be *strongly regular* if every its principal right (left) ideal is generated by a central idempotent. A module  $X$  is said to be *injective relative to the module  $Y$*  or  *$Y$ -injective* if for any submodule  $Y_1$  of  $Y$ , every homomorphism  $Y_1 \rightarrow X$  can be extended to a homomorphism  $Y \rightarrow X$ . A module is said to be *injective* if it is injective with respect to any module. A module is said to be *square-free* if it does not contain a direct sum of two non-zero isomorphic submodules. A submodule  $Y$  of the module  $X$  is said to be *essential* in  $X$  if  $Y \cap Z \neq 0$  for any non-zero submodule  $Z$  of  $X$ . A submodule  $Y$  of the module  $X$  is said to be *closed* in  $X$  if  $Y = Y'$  for every submodule  $Y'$  of  $X$  which is an essential extension of the module  $Y$ . We denote by  $\text{Sing } X$  the *singular submodule* of the right  $A$ -module  $X$ , i.e.,  $\text{Sing } X$  is a fully invariant submodule of  $X$  which consists of all elements  $x \in X$  such that  $r(x)$  is an essential right ideal of the ring  $A$ . A module  $X$  is said to be *non-singular* if  $\text{Sing } X = 0$ .

**Remark 1.1.** In [3, Theorem 7, Theorem 8, Example 9] Er, Singh and Srivastava proved the following results.

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- (1) If  $A$  is a right non-singular, right automorphism-invariant ring, then  $A = S \times T$ , where the ring  $S$  is right injective, the module  $T_T$  is square-free, any sum of closed right ideals of  $T$  is a two-sided ideal which is an automorphism-invariant right  $T$ -module, and for any prime ideal  $P$  of  $T$  which is not essential in  $T_T$ , the factor ring  $T/P$  is a division ring.
- (2) If  $A$  is a right non-singular, right automorphism-invariant prime ring, then the ring  $A$  is right injective.
- (3) Let  $F$  be the field of order 2,  $S$  the direct product of a countable set of copies of  $F$ , and  $A = \{(f_n)_{n=1}^\infty \in S : \text{almost all } f_n \text{ are equal to some } a \in F\}$ . Then  $A$  is a commutative automorphism-invariant regular ring, but it is not an injective  $A$ -module.

In connection to Remark 1.1, we will prove Theorem 1.2 which is the first main result of the given paper.

**Theorem 1.2.** *For a ring  $A$ , the following conditions are equivalent.*

- 1)  $A$  is a right automorphism-invariant right non-singular ring.
- 2)  $A$  is a right automorphism-invariant regular ring.
- 3)  $A = S \times T$ , where  $S$  is a right injective regular ring and  $T$  is a strongly regular ring which contains all invertible elements of its maximal right ring of quotients.

**Remark 1.3.** A module  $X$  is said to be *quasi-injective* if  $X$  is injective relative to  $X$ , i.e., for any submodule  $X_1$  of  $X$ , every homomorphism  $X_1 \rightarrow X$  can be extended to an endomorphism of the module  $X$ . Every quasi-injective module is an automorphism-invariant module, since the module  $X$  is quasi-injective if and only if  $X$  is invariant under any endomorphism of its injective hull; e.g., see [10, Theorem 6.74]. Every finite cyclic group is a quasi-injective non-injective module over the ring  $\mathbb{Z}$  of integers.

For a module  $X$ , we denote by  $G(X)$  the intersection of all submodules  $Y$  of the module  $X$  such that the factor module  $X/Y$  is non-singular. The submodule  $G(X)$  is a fully invariant submodule of  $X$ ; it is called the *Goldie radical* of the module  $X$ .

**Remark 1.4.** In [9, Theorem 3.8] Kutami and Oshiro proved that any direct sum of non-singular quasi-injective right modules over the ring  $A$  is quasi-injective if and only if  $A/G(A_A)$  is a semiprime right Goldie ring.

In connection to Remark 1.4, we will prove Theorem 1.5 which is the second main result of the given paper.

**Theorem 1.5.** *For a ring  $A$  with right Goldie radical  $G(A_A)$ , the following conditions are equivalent.*

- 1)  $A/G(A_A)$  is a semiprime right Goldie ring.
- 2) Any direct sum of automorphism-invariant non-singular right  $A$ -modules is an automorphism-invariant module.
- 3) Any direct sum of automorphism-invariant non-singular right  $A$ -modules is an injective module.

The proof of Theorem 1.2 and Theorem 1.5 is decomposed into a series of assertions, some of which are of independent interest.

We give some necessary definitions. A module  $X$  is said to be *singular* if  $X = \text{Sing } X$ . A module  $X$  is a *Goldie-radical* module if  $X = G(X)$ . The relation  $G(X) = 0$  is equivalent to the property that the module  $M$  is non-singular. In the paper, we use well-known properties of  $\text{Sing } X$ ,  $G(X)$ , non-singular modules and maximal right rings of quotients; e.g., see [4, Chapter 2], [10, Section 7] and [12, Section 3.3]. A module  $Q$  is called an *injective hull of the module  $M$*  if  $Q$  is an injective module and  $M$  is an essential submodule of the module  $Q$ . A module  $M$  is called a *CS module* if every its closed submodule is a direct summand of the module  $M$ . A module  $M$  is said to be *uniform* if the intersection of any two non-zero submodules of the module  $M$  is not equal to zero. A module  $M$  is said to be *finite-dimensional* if  $M$  does not contain an infinite direct sum of non-zero submodules.

A ring  $A$  is called a *right Goldie ring* if  $A$  is a right finite-dimensional ring with the maximum condition on right annihilators. A ring  $A$  is said to be *reduced* if  $A$  does not have non-zero nilpotent elements. A ring without non-zero nilpotent ideals is said to be *semiprime* ring. A ring  $A$  is said to be *right strongly semiprime* [6] if any its ideal, which is an essential right ideal, contains a finite subset with zero right annihilator. A ring is said to be *right strongly prime* [7] if every its non-zero ideal contains a finite subset with zero right annihilator.

**Remark 1.6.** Every right strongly semiprime ring is a right non-singular semiprime ring [6]. It is clear that every right strongly prime ring is right strongly semiprime. The direct product of two finite fields is a finite commutative strongly semiprime ring which is not strongly prime. The direct product of a countable number of fields is an example of a commutative semiprime non-singular ring which is not strongly semiprime. All finite direct products of rings without zero-divisors and all finite direct products of simple rings are right and left strongly semiprime rings.

**Remark 1.7.** If  $A$  is a semiprime right Goldie ring, then it is well known<sup>2</sup> that every essential right ideal of the ring  $A$  contains a non-zero-divisor. Therefore, all semiprime right Goldie rings are right strongly semiprime. In particular, all right Noetherian semiprime rings are right strongly semiprime.

## 2. Automorphism-Invariant Nonsingular Rings

**Lemma 2.1** [12, Section 3.3.]. *Let  $A$  be a right non-singular ring with maximal right ring of quotients  $Q$ . Then  $Q$  is an injective right regular ring and  $Q$  can be naturally identified with the ring  $\text{End } Q_A$  and  $Q_A$  is an injective hull of the module  $A_A$ .*

**Lemma 2.2.** *If  $A$  is a right non-singular ring with maximal right ring of quotients  $Q$ , then  $A$  is a right automorphism-invariant ring if and only if  $A$  contains all invertible elements of the ring  $Q$ .*

Lemma 2.2 follows from Lemma 2.1.

**Lemma 2.3** [?, Chapter 12, 5.1–5.4.]. *Let  $A$  be a reduced ring with*

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<sup>2</sup>For example, see [12, Theorem 3.2.14].

maximal right ring of quotients  $Q$ . Then ring  $A$  is right and left non-singular. If each of the closed right ideals of the ring  $A$  is an ideal, then  $A$  is a reduced ring and  $Q$  is a right and left injective strongly regular ring.

**Lemma 2.4.** *Let  $A$  be a right non-singular ring in which all closed right ideals are ideals. Then the ring  $A$  is reduced.*

**Proof.** Let  $a$  be an element of  $A$  with  $a^2 = 0$ . There exists a closed right ideal  $B$  of  $A$  such that  $B \cap aA = 0$  and  $B + aA$  is an essential right ideal of  $A$ . By assumption, the closed right ideal  $B$  is an ideal. Therefore,  $aB = 0$ , since  $aB$  is contained in the intersection of  $aA$  and  $B$ . Then  $a(B + aA) = 0$  and  $B + aA$  is an essential right ideal. Since  $A$  is right non-singular,  $a = 0$ .  $\square$

**Lemma 2.5.** *If  $A$  is a right automorphism-invariant right non-singular ring, then  $A = S \times T$ , where  $S$  is a right injective regular ring and  $T$  is a strongly regular ring which contains all invertible elements of its maximal right ring of quotients.*

**Proof.** By Remark 1.1(1) and Lemma 2.1,  $A = S \times T$ , where  $S$  is a right injective regular ring,  $T$  is a right automorphism-invariant right non-singular ring, and any closed right ideal of  $T$  is an ideal. By Lemma 2.4,  $T$  is a reduced ring. Let  $Q$  be the maximal right ring of quotients of the ring  $T$ . By Lemma 2.3,  $T$  is a reduced ring and  $Q$  is a right and left injective strongly regular ring. To prove that  $T$  is a strongly regular ring, it is sufficient to prove that an arbitrary element  $t$  of the ring  $T$  is the product of a central idempotent and an invertible element. Since  $t$  is an element of the strongly regular ring  $Q$ , we have that  $t = eu$ , where  $e$  is a central idempotent of the ring  $Q$  and  $u$  is an invertible element of the ring  $Q$ . By Lemma 2.2,  $T$  contains all invertible elements of the ring  $Q$ . Therefore,  $u \in T$ . Then  $e = tu^{-1} \in T$  and every element of the ring  $T$  is the product of a central idempotent and an invertible element.  $\square$

**Remark 2.6. The completion of the proof of Theorem 1.2.** In Theorem 1.2, the implication  $1) \Rightarrow 3)$  follows from Lemma 2.5, the implication  $3) \Rightarrow 2)$  follows from the property that the direct product of regular rings  $S$  and  $T$  is a regular ring, and the implication  $2) \Rightarrow 1)$  follows from the property that every regular ring is right and left non-singular.

**Corollary 2.7.** *If  $A$  is a right automorphism-invariant right non-singular indecomposable ring, then  $A$  is a right injective ring; see Remark 1.1(2).*

Corollary 2.7 follows from Theorem 1.2 and the property that every strongly regular indecomposable ring is a division ring and, consequently, a right injective ring.

**Corollary 2.8.** *Let  $A$  be a right automorphism-invariant right non-singular ring which does not contain an infinite set of non-zero central orthogonal idempotents. Then  $A$  is a right injective ring.*

Corollary 2.8 follows from Corollary 2.7 and the property that every ring, which does not contain an infinite set of non-zero central orthogonal idempotents, is a finite direct product of indecomposable rings.

### 3. Automorphism-Invariant Non-singular Modules

**Lemma 3.1.** *Let  $A$  be a ring and  $X$  a right  $A$ -module which is not an essential extension of a singular module. Then there exists a non-zero right*

ideal  $B$  of the ring  $A$  such that the module  $B_A$  is isomorphic to a submodule of the module  $X$ .

**Proof.** Since the module  $X$  is not an essential extension of a singular module, there exists an element  $x \in X$  such that  $xA$  is a non-zero non-singular module. Since  $xA \cong A_A/r(x)$  and the module  $xA$  is non-singular, the right ideal  $r(x)$  is not an essential. Therefore, there exists a non-zero right ideal  $B$  with  $B \cap r(x) = 0$ . In addition, there exists an epimorphism  $f: A_A \rightarrow xA$  with kernel  $r(x)$ . Since  $B \cap \text{Ker } f = 0$ , we have that  $f$  induces the monomorphism  $g: B \rightarrow xA$ . Therefore,  $xA$  contains the non-zero submodule  $g(B)$  which is isomorphic to the module  $B_A$ .  $\square$

**Lemma 3.2.** *Let  $A$  be a ring,  $G = G(A_A)$  the right Goldie radical of the ring  $A$ ,  $h: A \rightarrow A/G$  the natural ring epimorphism and  $X$  a non-singular non-zero right  $A$ -module.*

- 1) *If  $B$  is a essential right ideal of the ring  $A$ , then  $h(B)$  is an essential right ideal of the ring  $h(A)$ .*
- 2) *If  $B$  is a right ideal of the ring  $A$  such that  $G \subseteq B$  and  $h(B)$  is an essential right ideal of the ring  $h(A)$ , then  $B$  is an essential right ideal of the ring  $A$ .*
- 3) *For any right  $A$ -module  $M$ , the module  $MG$  is contained in the Goldie radical of  $M$ .*
- 4)  *$XG = 0$  and the natural  $h(A)$ -module  $X$  is non-singular. In addition, if  $Y$  is an arbitrary non-singular right  $A$ -module, then  $YG = 0$  and the  $h(A)$ -module homomorphisms  $Y \rightarrow X$  coincide with the  $A$ -module homomorphisms  $Y \rightarrow X$ . Therefore,  $X$  is an  $Y$ -injective  $A$ -module if and only if  $X$  is an  $Y$ -injective  $h(A)$ -module. The essential submodules of the  $h(A)$ -module  $X$  coincide with the essential submodules of the  $A$ -module  $X$ .*
- 5)  *$X$  is an injective  $h(A)$ -module if and only if  $X$  is an injective  $A$ -module.*
- 6)  *$X_{h(A)}$  is a uniform module (resp., an essential extension of a direct sum of uniform modules) if and only if  $X_A$  is a uniform module (resp., an essential extension of a direct sum of uniform modules).*
- 7)  *$X_A$  is an essential extension of a direct sum of modules each of them is isomorphic to some non-zero right ideal of the ring  $A$ .*
- 8) *If the ring  $A$  is right finite-dimensional, then  $X_A$  is an essential extension of a direct sum of modules each of them is isomorphic to some non-zero uniform right ideal of the ring  $A$ .*
- 9) *If the ring  $h(A)$  is right finite-dimensional, then  $X_{h(A)}$  is an essential extension of a direct sum of modules each of them is isomorphic to some non-zero uniform right ideal of the ring  $h(A)$ .*

**Proof. 1.** Let us assume that  $h(B)$  is not an essential right ideal of the ring  $h(A)$ . Then there exists a right ideal  $C$  of the ring  $A$  such that  $C$  properly

contains  $G$  and  $h(B) \cap h(C) = h(0)$ . Since  $h(B) \cap h(C) = h(0)$ , we have that  $B \cap C \subseteq G$ . Since  $C$  properly contains the closed right ideal  $G$ , we have that  $C_A$  contains a non-zero submodule  $D$  with  $D \cap G = 0$ . Since  $B$  is an essential right ideal,  $B \cap D \neq 0$  and  $(B \cap D) \cap G = 0$ . Then  $h(0) \neq h(B \cap D) \subseteq h(B) \cap h(C) = h(0)$ . This is a contradiction.

**2.** Let us assume that  $B$  is not an essential right ideal of the ring  $A$ . Then  $B \cap C = 0$  for some non-zero right ideal  $C$  of the ring  $A$  and  $G \cap C \subseteq B \cap C = 0$ . Therefore,  $h(C) \neq h(0)$ . Since  $h(B)$  is an essential right ideal of the ring  $h(A)$ , we have that  $h(B) \cap h(C) \neq h(0)$ . Let  $h(0) \neq h(b) = h(c) \in h(B) \cap h(C)$ , where  $b \in B$  and  $c \in C$ . Then  $c - b \in G \subseteq B$ . Therefore,  $c \in B \cap C = 0$  and  $h(c) = h(0)$ . This is a contradiction.

**3.** For any element  $m \in M$ , the module  $mG_A$  is a Goldie-radical module, since  $mG_A$  is a homomorphic image of the Goldie-radical module  $G$ . Therefore,  $mG \subseteq G(M)$  and  $MG \subseteq G(M)$ .

**4.** By 3,  $XG = 0$ . Let us assume that  $x \in X$  and  $xh(B) = 0$  for some essential right ideal  $h(B)$ , where  $B = h^{-1}(h(B))$  is the complete pre-image of  $h(B)$  in the ring  $A$ . By 2),  $B$  is an essential right ideal of the ring  $A$ . Then  $xB = 0$  and  $x \in \text{Sing } X = 0$ . Therefore,  $X$  is a non-singular  $h(A)$ -module. The remaining part of 4 is directly verified.

**5.** Let  $R$  be one of the rings  $A$ ,  $h(A)$  and  $M$  a right  $R$ -module. By Lemma 1(4), the module  $M$  is injective if and only if  $M$  is injective relative to the module  $R_R$ . Now the assertion follows from 4.

**6.** The assertion follows from 4.

**7.** Let  $\mathcal{M}$  be the set of all submodules of the module  $X$  which are direct sums of modules each of them is isomorphic to a non-zero right ideal of the ring  $A$ . The set  $\mathcal{M}$  is not empty by Lemma 3.1. There exists a partial order in  $\mathcal{M}$  such that for any  $M, M' \in \mathcal{M}$ , the relation  $M \not\subseteq M'$  is equivalent to the property that  $M' = M \oplus N$  for some  $N \in \mathcal{M}$ . By the Zorn lemma, the set  $\mathcal{M}$  contains at least one maximal element  $K$ .

Let us assume that  $K$  is not an essential submodule of the module  $X$ . Then there exists a non-zero submodule  $L$  of the non-singular module  $X$  with  $K \cap L = 0$ . By Lemma 3.1, there exists a non-zero right ideal  $B$  of the ring  $A$  such that the module  $B_A$  is isomorphic to some submodule  $L'$  of the module  $L$ . This contradicts to the property that  $K$  is a maximal element of the set  $\mathcal{M}$ .

**8.** Since the ring  $A$  is right finite-dimensional, every non-zero right ideal of the ring  $A$  is an essential extension of a finite direct sum of non-zero uniform right ideals. Now the assertion follows from 7.

**9.** The assertion follows from 6 and 8. □

**Remark 3.3.** Let  $M$  be an automorphism-invariant non-singular module. In [3, Theorem 3, Theorem 6(ii)] Er, Singh and Srivastava proved that  $M = X \oplus Y$ , where  $X$  is a quasi-injective non-singular module,  $Y$  is an automorphism-invariant non-singular square-free module, the modules  $X$  and  $Y$  are injective relative to each other,  $\text{Hom}(X, Y) = 0 = \text{Hom}(Y, X)$  and  $\text{Hom}(D_1, D_2) = 0$  for any two submodules  $D_1, D_2$  of the module  $Y$  with  $D_1 \cap D_2 = 0$ . In addition, for any set  $\{K_i \mid i \in I\}$  of closed submodules of  $Y$ , the submodule  $\sum_{i \in I} K_i$  is

an automorphism-invariant module.

**Remark 3.4.** Let  $M$  be a direct sum of CS modules  $M_i$ ,  $i \in I$ . In [11, Corollary 15] Lee and Zhou proved that  $M$  is a quasi-injective module if and only if  $M$  is an automorphism-invariant module.

**Remark 3.5.** Let  $A$  be a ring with right Goldie radical  $G(A_A)$ . In [9, Theorem 3.4] Kutami and Oshiro proved that the factor ring  $A/G(A_A)$  is a right strongly semiprime ring if and only if every non-singular quasi-injective right  $A$ -module is injective.

In [9, Theorem 3.8] Kutami and Oshiro proved that the factor ring  $A/G(A_A)$  is a semiprime right Goldie ring if and only if every direct sum of non-singular quasi-injective right  $A$ -modules is quasi-injective.

**Lemma 3.6.** *Let  $A$  be a ring with right Goldie radical  $G(A_A)$  and  $M$  an automorphism-invariant non-singular right  $A$ -module which is an essential extension of a direct sum of uniform modules.*

- 1)  *$M$  is an essential extension of some quasi-injective non-singular module  $K$  which is direct sum of uniform modules closed in  $M$ .*
- 2) *If the factor ring  $A/G(A_A)$  is a right strongly semiprime ring, then  $M$  is an injective module.*

**Proof. 1.** By Remark 3.3,  $M = X \oplus Y$ , where  $X$  is a quasi-injective module,  $Y$  is an automorphism-invariant square-free module. Therefore, we can assume that  $M$  is an automorphism-invariant square-free module. Since  $M$  is an essential extension direct sum of uniform submodules,  $M$  is an essential extension of some module  $K$  which is the direct sum of uniform closed submodules  $K_i$  of  $M$ ,  $i \in I$ . By Remark 3.3,  $K$  is an automorphism-invariant module. Since every uniform module is a CS module,  $K$  is a quasi-injective module by Remark 3.4.

**2.** By 1,  $M$  is an essential extension of some quasi-injective non-singular module  $K$ . By Remark 3.5,  $K$  is an injective essential submodule of the module  $M$ . Therefore,  $K$  is an essential direct summand of the module  $M$ . Then  $M = K$  and  $M$  is an injective module.  $\square$

**Lemma 3.7.** *Let  $A$  be a ring with right Goldie radical  $G(A_A)$  and  $M$  an automorphism-invariant non-singular right  $A$ -module. If the factor ring  $A/G(A_A)$  is a semiprime right Goldie ring, then  $M$  is an injective module.*

**Proof.** By Lemma 3.2(9),  $M$  is an essential extension of a direct sum of uniform modules. In addition, the semiprime right Goldie ring  $A/G(A_A)$  is a right strongly semiprime ring [6]. By Lemma 3.6(2),  $M$  is an injective module.  $\square$

**Remark 3.8. The completion of the proof of Theorem 1.5.** In Theorem 1.5, the implications  $3) \Rightarrow 2) \Rightarrow 1)$  are obvious.

$1) \Rightarrow 3)$ . Let  $M$  be the direct sum of automorphism-invariant non-singular right  $A$ -modules  $M_i$ ,  $i \in I$ . By Lemma 3.7, each of the modules  $M_i$  is injective. By Remark 3.5,  $M$  is a quasi-injective module. By Remark 3.5,  $M$  is an injective module.  $\square$

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